Last Time

- Asymptotic Notation
- Review of Proofs by Induction
Recursion
  - Binary Search
  - MergeSort
  - QuickSort
A **recursive** function calls itself.
A recursive function calls itself.

Requirements to avoid infinite loops
- A stopping condition
- Modification of input
Fibonacci Series

- 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

- Related to the Golden Ratio $\phi$
Formula for Calculating Fibonacci numbers

- \( Fib(1) = 1 \)
- \( Fib(2) = 1 \)
- \( Fib(n) = Fib(n - 1) + Fib(n - 2) \) for \( n > 2 \)

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots
Fib(n) in pseudocode

Fib(n)

1: if n = 1 then
2:   return 1
3: end if
4: if n = 2 then
5:   return 1
6: end if
7: return Fib(n-1) + Fib(n-2)
Runtime of $Fib(n)$

- How many times is $Fib(n)$ called?
Fib(3)

1

Fib(2)  Fib(1)

1  1

2
Fib(4)

```
Fib(4)
  /    \
Fib(3)  Fib(2)
/  \      \  
Fib(2)  Fib(1)  1
   \    \  
    1    1
    1    2
```
Fib(n)

1 = 2^0
2 = 2^1
4 = 2^2
8 = 2^3
Fib(n) = O(2^n)

We will return to Fib(n) when we talk about Dynamic Programming.
\[ Fib(n) = O(2^n) \]

- To think about. \( Fib(n) = \Theta(2^{\frac{n}{2}}) \)

- We will return to \( Fib(n) \) when we talk about Dynamic Programming.
Find the index $i$ of an item $x$ in a sorted array $A$ of size $n$.

```
int Find(A, x)
1: ...
2: return i
```
Sequential Search

```plaintext
int Find(A, x)

1:  for i ← 0..N-1 do
2:      if A[i] = x then
3:          return i
4:      end if
5:  end for
6:  return Not Found
```
The for loop (line 1-5) runs N times.

Best Case $A[0] = x \Theta(1)$

Worst Case $A[n] = x$ or $x$ is not in $A \Theta(n)$

Expected Case if $x$ is in $A$, $\Theta(n^2) = \Theta(n)$, if not, $\Theta(n)$

```plaintext
int Find(A, x)

1:    for i ← 0..N-1 do
2:        if A[i] = x then
3:            return i
4:        end if
5:    end for
6:    return Not Found
```
Binary Search Example

- $A = [10, 13, 14, 29, 37]$
- Find 29 in $A$
- Find 29 in $A_2$
- Check 29. Success.
int Find(A, x)

1: return BinarySearch(A, x, 1, N+1)

int BinarySearch(A, x, low, high)

1: if low > high then
2: return Not Found
3: end if
4: mid ← \([\frac{low+high}{2}]\)
5: if x = A[mid] then
6: return mid
7: end if
8: if x < [mid] then
9: return BinarySearch(A, x, low, mid-1)
10: else
11: return BinarySearch(A, x, mid+1, high)
12: end if
Logarithmic and Exponential Growth

- Note: These are crude rules of thumb.
- Multiplication at each recursive step leads to exponential growth
- Division at each recursive step leads to logarithmic growth
Each recursive call to BinarySearch cuts the examined size of $A$ in half.

Therefore suspect that $\text{BinarySearch}(A, x, \text{low, high})$ is $\Theta(\log n)$

**Best Case** $A[\text{mid}] = x \Theta(1)$

**Worst Case** $x$ is not in $A \Theta(\log n)$

**Expected Case** if $x$ is in $A, \Theta(\log n - 1) = \Theta(\log n)$. $x$ is not in $A \Theta(\log n)$
Binary vs. Sequential Search

- Binary Search - $\Theta(\log n)$
- Sequential Search - $\Theta(n)$
- ...but Binary search requires $A$ to be sorted.
- The best sorting we’ve looked at is $\Theta(n^2)$. Can we do better?
Divide and Conquer

- A recursive strategy.
- **Divide** the problem into subproblems
- **Conquer** the subproblems recursively
- **Combine** the subproblem solutions into a solution of the initial problem.
To construct a sorted list from two sorted lists is $\Theta(n)$.

- $A_1 = [1, 4, 7, 9, 11]$
- $A_2 = [2, 3, 8, 10]$
- $A = []$
• To construct a sorted list from two sorted lists is $\Theta(n)$.
• $A_1 = [4, 7, 9, 11]$
• $A_2 = [2, 3, 8, 10]$
• $A = [1]$
To construct a sorted list from two sorted lists is $\Theta(n)$.

- $A_1 = [4, 7, 9, 11]$
- $A_2 = [3, 8, 10]$
- $A = [1, 2]$
Merge Sort

- To construct a sorted list from two sorted lists is $\Theta(n)$.
- $A_1 = [4, 7, 9, 11]$
- $A_2 = [8, 10]$
- $A = [1, 2, 3]$
To construct a sorted list from two sorted lists is $\Theta(n)$.

- $A_1 = [7, 9, 11]$
- $A_2 = [8, 10]$
- $A = [1, 2, 3, 4]$
To construct a sorted list from two sorted lists is $\Theta(n)$.

- $A_1 = [9, 11]
- A_2 = [8, 10]
- A = [1, 2, 3, 4, 7]
To construct a sorted list from two sorted lists is $\Theta(n)$.

- $A_1 = [9, 11]$
- $A_2 = [10]$
- $A = [1, 2, 3, 4, 7, 8]$
To construct a sorted list from two sorted lists is $\Theta(n)$.

- $A_1 = [11]$
- $A_2 = [10]$
- $A = [1, 2, 3, 4, 7, 8, 9]$
To construct a sorted list from two sorted lists is $\Theta(n)$.

- $A_1 = [11]$
- $A_2 = []$
- $A = [1, 2, 3, 4, 7, 8, 9, 10]$
To construct a sorted list from two sorted lists is $\Theta(n)$.

- $A_1 = []$
- $A_2 = []$
- $A = [1, 2, 3, 4, 7, 8, 9, 10, 11]$
Apply Divide and Conquer to Merge Sort

- Divide the $n$ element array into two subarrays with $\frac{n}{2}$ elements each to be sorted.
- Conquer by sorting the subsequences using Merge Sort
- Combine the two subsequences (as above) to construct the sorted array
MergeSort(A)

if size(A) = 1 then
    return A
end if

mid ← ⌊size(A)/2⌋
A_1 = MergeSort(A[1..mid])
A_2 = MergeSort(A[mid+1..N])
return Merge(A_1, A_2)
Merge Sort Example

- Divide
- $A = [10, 2, 6, 4]$
Merge Sort Example

- **Divide**
- \( A = [10, 2, 6, 4] \)
- \( A = [10, 2][6, 4] \)
Merge Sort Example

- Divide
- $A = [10, 2, 6, 4]$
- $A = [10, 2][6, 4]$
- $A = [10][2][6][4]$

- **Conquer** (or Merge)
- \(A = [10, 2, 6, 4]\)
- \(A = [2, 10][4, 6]\)
- \(A = [10][2][6][4]\)
Merge Sort Example

- **Conquer** (or Merge)
  - $A = [2, 4, 6, 10]$
  - $A = [2, 10][4, 6]$
  - $A = [10][2][6][4]$
How long does MergeSort take? $\Theta(?)$
Runtime analysis of MergeSort

- How long does MergeSort take? $\Theta(?)$
- How many levels of recursion are there?
Runtime analysis of MergeSort

- How long does MergeSort take? $\Theta(\,?)$
- How many levels of recursion are there?
- $\log n$
Runtime analysis of MergeSort

- How long does MergeSort take? $\Theta(\cdot)$
- How many levels of recursion are there? $\log n$
- How many operations are performed at each level of the recursion?
Runtime analysis of MergeSort

- How long does MergeSort take? $\Theta(?)$
- How many levels of recursion are there? $\log n$
- How many operations are performed at each level of the recursion? $n$
Runtime analysis of MergeSort

- How long does MergeSort take? $\Theta(\cdot)$
- How many levels of recursion are there? $\log n$
- How many operations are performed at each level of the recursion? $n$
- Merge Sort is $\Theta(n \log n)$
Quick Sort

- Another recursive sorting Algorithm
- Concatenating arrays is $\Theta(1)$
- Identifying those points in an array that are higher than a point is $\Theta(n)$
Quick Sort

- Identify a point $A[i]$ from the $A$ called the “pivot”
- Construct lists $A_1$ and $A_2$ such that $A_1$ contains all the elements of $A$ less than $A[i]$, and $A_2$ those greater than $A[i]$
- Recursively sort $A_1$ and $A_2$
- Concatenate $A_1$, $A[i]$ and $A_2$
Quick Sort Example

- Divide
- $A = [10, 2, 6, 4]$
Quick Sort Example

- Divide
- \( A = [10, 2, 6, 4] \)
- Select pivot 6
Quick Sort Example

- Divide
- A = [10, 2, 6, 4]
- Select pivot 6
- A = [2, 4][6][10]
Quick Sort Example

- Divide
- A = [10, 2, 6, 4]
- Select pivot 6
- A = [2, 4][6][10]
- A₁ = [2, 4]
Quick Sort Example

- **Divide**
- $A = [10, 2, 6, 4]$
- Select pivot 6
- $A = [2, 4][6][10]$
- $A_1 = [2, 4]$
- Select pivot 4
Quick Sort Example

- Divide
- $A = [10, 2, 6, 4]$
- Select pivot 6
- $A = [2, 4][6][10]$
- $A_1 = [2, 4]$
- Select pivot 4
- $A = [2][4]$
Quick Sort Example

- Combine

Quick Sort Example

- Combine
- $A = [2, 4][6][10]$ return $[2, 4, 6, 10]$
How long does QuickSort take? $\Theta(\cdot)$
Runtime analysis of QuickSort

- How long does QuickSort take? \( \Theta(?) \)
- How many levels of recursion are there?
Runtime analysis of QuickSort

- How long does QuickSort take? $\Theta(\cdot)$
- How many levels of recursion are there?
- $\log n$
Runtime analysis of QuickSort

- How long does QuickSort take? \( \Theta(?) \)
- How many levels of recursion are there?
  \( \log n \)
- How many operations are performed at each level of the recursion?
Runtime analysis of QuickSort

- How long does QuickSort take? $\Theta(?)$
- How many levels of recursion are there?
  - $\log n$
- How many operations are performed at each level of the recursion?
  - $n$
How long does QuickSort take? $\Theta(\cdot)$

How many levels of recursion are there?

$\log n$

How many operations are performed at each level of the recursion?

$n$

Quick Sort is $\Theta(n \log n)$
## QuickSort vs. MergeSort

<table>
<thead>
<tr>
<th></th>
<th>MergeSort</th>
<th>QuickSort</th>
</tr>
</thead>
<tbody>
<tr>
<td>Divide</td>
<td>$\Theta(1)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>Conquer</td>
<td>$\Theta(\log n)$</td>
<td>$\Theta(\log n)$</td>
</tr>
<tr>
<td>Combine</td>
<td>$\Theta(n)$</td>
<td>$\Theta(1)$</td>
</tr>
</tbody>
</table>
Expected vs. Worst Case runtime

- Quicksort has an expected runtime of $\Theta(n \log n)$
- QuickSort could be as slow as $\Theta(n^2)$
Expected vs. Worst Case runtime

- Quicksort has an expected runtime of $\Theta(n \log n)$
- QuickSort could be as slow as $\Theta(n^2)$
- How?
Bye

Next time (9/16)

- Analyzing the runtime of general recursive algorithms.
- Read Chapter 8.