Lecture 10: Balanced Binary Search Trees
CSCI 700 - Algorithms I

Andrew Rosenberg
Last Time

- Balanced Binary Search Trees – AVL Trees
Today

- More Balanced Binary Search Trees
  - Red-Black Trees
  - 2-3 Trees
  - B-Trees
AVL Trees are **Binary Search Trees**

- Maintain near perfect balance on insert and delete
- Rotation operations
- Require storage at each node of balance factor $bf \in \{-1, 0, 1\}$ or height $h \in \mathbb{Z}$
Red-Black Trees are **Binary Search Trees**

In addition to **BST Properties**, they also satisfy **Red-Black Tree Properties (or RBT Properties)**

1. Every node is either red or black. (1-bit storage)
2. The root is black
3. Nulls (below leaves) are black.
4. If a node is red, all of its children are black.
5. For each node, all paths from the node to descendent leaves contain the same number of black nodes.
   - We’ll call this **black-height** of a node $bh(T)$. 
Example of a Red-Black Tree

$$h = 4$$
Example of a Red-Black Tree

```
7
 /    \
3     18
 |      |
NIL    NIL
```

```
10
 /    \
8     11
 |      |
NIL    NIL
```

```
22
 /    \
NIL    NIL
```

```
26
 /    \
NIL    NIL
```

bh values:
- 7: 2
- 3: 1
- 18: 2
- 10: 1
- 8: 1
- 11: 1
- 22: 0
- 26: 0
We can track the balance of the whole tree using only local information about the color of a node and its parent and children.

Color information is stored in a single bit.

Persistent data structures In AVL trees deletion may require up to $O(\log n)$ rotations. In R-B Trees it will require $O(1)$. Therefore to store rollback information requires $\log n$ times the space when implemented using an AVL tree.
A red-black tree with $n$ internal nodes has height at most \(2 \log (n + 1)\).

■ Show that a subtree, $T$, has at least \(2^{bh(T)} - 1\) internal nodes.
■ Induction on height of $T$.
■ Base case: If $T$.height == 0, then $T$ is a leaf. Therefore $T$ contains 0 internal nodes. \(2^{bh(T)} - 1 = 2^0 - 1 = 0\).
A red-black tree with $n$ internal nodes has height at most $2 \log (n + 1)$.

- Show that a subtree, $T$, has at least $2^{bh(T)} - 1$ internal nodes.
- Induction on height of $T$.
- Inductive step: $T$ has positive height, and is an internal node with 2 children. Each child has a black-height of $bh(T)$ (if the child is red) or $bh(T) - 1$ (if the child is black).
A red-black tree with $n$ internal nodes has height at most $2 \log (n + 1)$.

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- By induction, each child has at least $2^{bh(T) - 1} - 1$ internal nodes.
Theorem

A red-black tree with \( n \) internal nodes has height at most \( 2 \log (n + 1) \).

- Show that a subtree, \( T \), has at least \( 2^{bh(T)} - 1 \) internal nodes.
- Induction on height of \( T \).
- Inductive step: \( T \) has positive height, and is an internal node with 2 children. Each child has a black-height of \( bh(T) \) (if the child is red) or \( bh(T) - 1 \) (if the child is black).
- By induction, each child has at least \( 2^{bh(T)-1} - 1 \) internal nodes.
- Therefore, \( T \) has at least
\[
(2^{bh(T)-1} - 1) + (2^{bh(T)-1} - 1) + 1 = 2^{bh(T)} - 1
\] internal nodes.
Theorem

A red-black tree with \( n \) internal nodes has height at most \( 2 \log (n + 1) \).

- Show that a subtree, \( T \), has at least \( 2^{bh(T)} - 1 \) internal nodes.
- Let \( h \) be the height of \( T \).
Theorem

A red-black tree with $n$ internal nodes has height at most $2 \log (n + 1)$.

- Show that a subtree, $T$, has at least $2^{bh(T)} - 1$ internal nodes.
- Let $h$ be the height of $T$.
- At least half of the items on any path from $T$ to a leaf must be black (Property 4).
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- Thus \( bh(T) \geq h/2 \)
Height of a Red-Black Tree

**Theorem**

*A red-black tree with n internal nodes has height at most $2 \log (n + 1)$.*

- Show that a subtree, $T$, has at least $2^{bh(T)} - 1$ internal nodes.
- Let $h$ be the height of $T$.
- At least half of the items on any path from $T$ to a leaf must be black (Property 4).
- Thus $bh(T) \geq h/2$
- So....$n \geq 2^{h/2} - 1$
- $n + 1 \geq 2^{h/2}$.
- $\log (n + 1) \geq h/2$
- $2 \log (n + 1) \geq h$
A red-black tree with \( n \) internal nodes has height at most 
\[ 2 \log (n + 1). \]

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- So....\( n \geq 2^{h/2} - 1 \)
- \( n + 1 \geq 2^{h/2} \).
- \( \log (n + 1) \geq h/2 \)
- \( 2 \log (n + 1) \geq h \)
- Runtime in \( O(h) = O(\log n) \)
Search, Min, Max, Successor, and Predecessor all run in $O(h) = O(\log n)$ time, on a red-black tree with $n$ nodes.
The operation itself is unchanged.

May require color changes.

May require rotations to maintain Property 4 (If a node is red, it’s children are black).
Rotation Review

- Rotation maintains the **BST property**.
- Rotations take $O(1)$.
Red-Black Tree Insertion

- Idea: Insert $x$ into the tree $T$.
- Color $x$ red. – Thus $bh(T)$ is maintained for all subtrees that $x$ is a member of.
- However, Property 4 – If a node is red, all of its children are black – may not hold.
Red-Black Tree Insertion

- Idea: Insert $x$ in to the tree $T$.
- Color $x$ red. – Thus $bh(T)$ is maintained for all subtrees that $x$ is a member of.
- However, Property 4 – If a node is red, all of its children are black – may not hold
- Recolor and rotate until the **RBT Property** is restored.
Example:
Example:

- Insert \( x = 15 \).
- Recolor, moving the violation up the tree.
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- `RIGHT-ROTATE(18)`.
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- **RIGHT-ROTATE(18).**
- **LEFT-ROTATE(7)** and recolor.
Example:

- Insert $x = 15$.
- Recolor, moving the violation up the tree.
- **RIGHT-ROTATE(18)**.
- **LEFT-ROTATE(7)** and recolor.
Red-Black Insert Pseudocode

```plaintext
RB-Insert(T, x)
Insert(T, x)
  x.Color ← RED
while x ≠ T and x.parent.color = RED do
  if IsLeftChild(x.parent) then
    T.color ← BLACK
    y ← x.parent.parent.right
    if y.color = RED then
      Case 1
    else
      if IsRightChild(x) then
        Case 2
      end if
      Case 3
    end if
  else
    swap left and right
  end if
end while
```
Red-Black Case 1

(Or, children of $A$ are swapped.)

Push $C$’s black onto $A$ and $D$, and recurse, since $C$’s parent may be red.
Red-Black Case 2

\textbf{LEFT-ROTATE}(A)

Transform to Case 3.
Red-Black Case 3

RIGHT-ROTATE(C)

Done! No more violations of RB property 3 are possible.
Red-Black Insert Analysis

- First traverse up the tree recoloring.
- If Case 2 or 3 occurs, rotate once or twice.
Red-Black Insert Analysis

- First traverse up the tree recoloring.
- If Case 2 or 3 occurs, rotate once or twice.
- Runtime: $O(h) = O(\log n)$ and $O(1)$ rotations.

- Red-Black Delete has the same running time and number of rotations as insert. Refer to the text for this.
2-3 Trees are Search Trees where each node can have 1 or 2 keys and 2 or 3 children.
Example 2-3 Trees
Each leaf has the same depth and contains 1 or 2 keys.

Each interior node:
- contains 1 key and has 2 children (2-node)
- contains 2 key and has 3 children (3-node)

In a 2-node $T$ with key $a$
- each key in its left subtree has key $\leq a$
- each key in its right subtree has key $> a$

In a 3-node $T$ with keys $a$ and $b$
- each key in its left subtree has key $\leq a$
- each key in its middle subtree has $a < key \leq b$
- each key in its right subtree has key $> b$
What is the height, $h$ of a tree containing $n$ values?
What is the height, \( h \) of a tree containing \( n \) values?

- Each internal node can have up to 3 children.
- There are \( 3^{h-1} \) leaves.
What is the height, $h$ of a tree containing $n$ values?

Each internal node can have up to 3 children.

There are $3^{h-1}$ leaves.

Each leaf can have up to 2 values

$n \leq 2 \times 3^{h-1}$

$log_6 n - 1 \leq h$
What is the height, $h$, of a tree containing $n$ values?

Each internal node has at least two children.

There are at least $2^{h-1}$ leaves.
What is the height, $h$ of a tree containing $n$ values?
- Each internal node has at least to 2 children.
- There are at least $2^{h-1}$ leaves.
- Each leaf has at least 1 value
- $n \geq 1 \times 2^{h-1}$
- $\log_2 n - 1 \geq h$
What is the height, \( h \) of a tree containing \( n \) values?

- Each internal node has at least two children.
- There are at least \( 2^{h-1} \) leaves.
- Each leaf has at least 1 value

\[
n \geq 1 \times 2^{h-1}
\]
\[
\log_2 n - 1 \geq h
\]
\[
\log_2 n - 1 \geq h \geq \log_6 n - 1
\]
\[
h = \Theta(\log n)
\]
2-3 Tree Insert

- Find the leaf $l$ to insert $x$ as in BST Insert.
- If $l$ has 3 keys, move the middle key of $l$ up to its parent, $p$, and split $l$ into 2 leaves.
2-3 Tree Insert

- Find the leaf \( l \) to insert \( x \) as in BST Insert.
- if \( l \) has 3 keys, move the middle key of \( l \) up to its parent, \( p \), and split \( l \) into 2 leaves.
- while \( p \) has 3 keys (and 4 children)
  - split \( p \) into \( p_1 \) and \( p_2 \).
  - register \( p_1 \) and \( p_2 \) instead of just \( p \) as children of the parent of \( p \).
  - \( p \leftarrow p.\text{parent} \)
- if the root was split, insert a new root to hold \( p_1 \) and \( p_2 \)
2-3 Insert example
2-3 Insert example

Insert (26)
2-3 Insert example
2-3 Insert example
What is the runtime of 2-3 Insert?
What is the runtime of 2-3 Insert?

\( \Theta(\log n) \)
2-3 Delete

- Find the node $T$ containing $x$.
- If $T$ 2-leaf, delete $x$ from the leaf.
- Else, replace $x$ by $\text{Successor}(x)$
- This might make a node $w$ have no keys (illegal).
- While $w$ is illegal:
  - If $w$ has a sibling $w'$ with 2 keys, steal one of the keys.
  - If $w$ has a sibling $w'$ with 1 key, merge them.

- if $w$ is the root, delete $w$, and let $\text{root} \leftarrow w.\text{child}$. 
2-3 Delete

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- If $T$ 2-leaf, delete $x$ from the leaf.
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- This might make a node $w$ have no keys (illegal).
- While $w$ is illegal:
  - If $w$ has a sibling $w'$ with 2 keys, steal one of the keys.
    - Let $s$ be the key in the parent $u$ of $w$ and $w'$ separating them.
      move $s$ from $u$ to $w$, replace $s$ in $u$ by nearest key in $w'$.
  - If $w$ has a sibling $w'$ with 1 key, merge them.
- If $w$ is the root, delete $w$, and let $\text{root} \leftarrow w.\text{child}$.
Find the node $T$ containing $x$.

If $T$ is a 2-leaf, delete $x$ from the leaf.

Else, replace $x$ by $\text{Successor}(x)$

This might make a node $w$ have no keys (illegal).

While $w$ is illegal:

- If $w$ has a sibling $w'$ with 2 keys, steal one of the keys.
  - Let $s$ be the key in the parent $u$ of $w$ and $w'$ separating them.
    - move $s$ from $u$ to $w$, replace $s$ in $u$ by nearest key in $w'$.
- If $w$ has a sibling $w'$ with 1 key, merge them.
  - Merge $w$ and $w'$ to a new 3-node $w''$ with keys $s$ and that of $w'$.
  - $w \leftarrow \text{parent}(w)$ – may have become illegal.
- If $w$ is the root, delete $w$, and let $\text{root} \leftarrow w.\text{child}$. 
2-3 Delete examples

The left most key has just been deleted.
B-Trees

- **B-trees** are a generalization of 2-3 trees where each node has between B and 2B-1 children.
- Essentially an \((a,b)\)-tree, where \(b = 2a - 1\).
B-Trees

- **B-trees** are a generalization of 2-3 trees where each node has between $B$ and $2B-1$ children.
- Essentially an $(a,b)$-tree, where $b = 2a-1$.
- Disk based storage. Databases
- If each page can hold $2B$ records, this is an efficient use of disk reads.
Bye

- Next time
  - Heaps
- For Next Class
  - Read 16.1, 16.2, 16.3